The structure of all possible solutions to the Buckley-Leverett model (new).

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Abstract

We consider a free boundary problem for a one-dimensional system of Buckley-Leverett equations, describing the displacement of oil by a suspension. For this problem we formulated conditions for the strong decay of the discontinuity of the initial oil concentration. We will prove that the phenomenological Buckley-Leverett model does not adequately describe the physical process under consideration. To do this, we will study the problem of the decay of a discontinuity in the initial concentration of oil, when at rest in one half of the domain there is oil, and in the other half of the domain there is a suspension, and these domains are separated by an impenetrable partition. At the initial moment in time, the partition is removed, which initiates the movement of the fluids. A precise analysis of the unique solution to the corresponding initial-boundary value problem for the Buckley-Leverett model shows that there are several different configurations of movement for oil, suspension, and mixture, depending on the initial and boundary data.

We will prove that for all these configurations of initial and boundary data, the model describes unrealistic fluid motion. For example, the mixture begins to displace oil and suspension.

Key words: Free boundary problems, transport equations, displacement of oil by suspension, strong discontinuity conditions.

1 Introduction.

Free boundary problems for differential equations are some of the most difficult in the theory of partial differential equations. In these problems, along

with solving differential equations, it is necessary to determine the domain in which this solution is sought. As a rule, this domain (boundary) is determined from an additional boundary condition at the free boundary. In the theory of free boundary problems, the Stefan problem, the Masket problem, and the Hele-Shaw problem [1]–[5] for the heat or Laplace equations are well known. These problems are formulated quite simply, but so far the existence of a classical solution has been proven only locally in time (excluding some simple cases). As for systems of differential equations, here we should note the works of V. A. Solonnikov for free boundary problems to the Navier-Stokes system [6], [7] and A. Friedman [8].

Note that the one-phase Stefan problem has a time monotone free boundary, which allowed authors [1] to prove the existence of a classical solution for one-phase Stefan problem arbitrary time interval.

However, in all other cases, including two-phase only the local existence of a solution can be proven.

Separately, there is a large class of free boundary problems for the equations of gas dynamics and hydrodynamics of an ideal incompressible fluid. These problems are well studied and have a rich history [9]-[10].

We will consider the Buckley–Leverett model, formulated in [11], [12] and describing the displacement of oil by a suspension in the pore space of the absolutely rigid solid skeleton at the macroscopic level. Let us recall that such models as [12]-[14] are called ***Phenomenological***, which means that all these models are nothing more then some set of axioms (postulates).

Unlike ***Microscopic Mathematical Models***, with characteristic size is approximately tens of microns, in ***Macroscopic Mathematical Models*** the characteristic size is approximately meters or tens of meters. Because of this, these models do not distinguish between the microstructure of a continuous medium, since in such a model at each point the medium contains solid skeleton, fluids in the pores of this skeleton and ***Free (Unknown) Boundary***, separated these liquids.

All such models are built on the same principle. Fluid dynamics is usually controlled by the Darcy's system of filtration, or some modification of it. The equations describing the migration of oil and suspension simply postulated and, roughly speaking, are some modifications of continuity equations for the corresponding concentrations of oil and suspension. Finally, the difference of pressers in oil and in suspension is some known function of the oil concentration.

Exactly here there is a great variety of models, depending on the tastes and preferences of its authors. It is quite understandable, since the main mechanism of the physical process is focused on the unknown (free) boundary between different fluids and not spelled out in any way in the proposed macroscopic models.

The existence and uniqueness of a generalized solution to the system of

Buckley–Leverett equations for smooth data of the problem was proved by S. Antontsev and V. Monakhov [15]. We will be interested in the structure of the weak solution to the Buckley–Leverett system of equations for a discontinuous initial oil concentration. In the terminology of L. V. Ovsyannikov [16] (English translation see in Appendix A, section A6 in [5]), such a problem is called ***The problem of the decay of a strong discontinuity***. The Buckley–Leverett model and its analogs are phenomenological mathematical models and serve as the basis model for existing hydrodynamic simulators, such as ***Eclipse***, ***Black Oil*** (Schlumberger), ***Tempest*** (Roxar), ***VIP*** (Landmark) and ***TimeZYX*** (Standard Oil and Trust).

Note, that a hydrodynamic simulator is a certain ***Scale*** (set) of mathematical models of an oil reservoir of varying degrees of accuracy, supplemented with digital characteristics of physical properties, such as density and elastic properties of the solid skeleton, soil, density and viscosity of filtered liquids, as well as geometric characteristics of the reservoir in consideration, such as the structure of the solid skeleton, the geometry of the domain occupied by the reservoir, and visualization programs for numerical implementations.

Existing simulators, according to their purpose, must adequately reflect the simulated physical process.

Do existing simulators solve this problem?

Let's reformulate the question differently.

Since the basis of any hydrodynamic simulator is the corresponding scale of mathematical models (ideally!), the question can be formulated as follows:

Do the existing mathematical models underlying existing hydrodynamic simulators adequately reflect the physical process being modeled?

Only adequate mathematical models can optimize the oil production process, and only with adequate modeling can the main problem of a hydrodynamic simulator be solved - this is, of course, obtaining ***Maximum benefit from the exploitation of the field***.

A positive or negative answer depends on what exactly ***Adequacy of a mathematical model for a given physical process means.***

To do this, it is necessary to formulate ***Adequacy Criteria*** of the mathematical model.

In the case of phenomenological models, the criterion of adequacy can only be experiment. Is experiment a criterion of adequacy?

The answer is NO.

In fact, it makes no sense to talk about an experiment, since in any phenomenological model there are enough free parameters and even functions that are in no way related to the geometry of the reservoir (porosity and structure of the pore space) or to the physical characteristics of the displacement process (viscosity and density of filtered liquids and density and

elastic properties of the solid skeleton). Therefore, by varying the indicated constants and functions, one can achieve agreement with any experiment!

Let us recall that, by definition, any phenomenological mathematical model is a set of postulates (axioms) expressed using differential equations, supplemented by appropriate boundary and initial conditions, as well as defining relations.

In this case, the characteristic dimensions in macroscopic models are meters or tens of meters. Because of this, such models do not distinguish between the microstructure of the continuous medium, nor the free boundary separating the liquids, nor the features of the interaction of liquids with the solid skeleton of the soil (adhesion or sliding conditions), since in such a model at each point of the continuous medium there are rock (solid skeleton) together with liquid in the pores of this skeleton, and a free boundary separating the various components of the medium.

All such models are built on the same principle. The dynamics of the fluid are usually controlled by the Darcy filtration equations or some modification of it, and the interaction of fluids is regulated by the laws of conservation of mass for each fluid. But all the fundamentally important changes occur precisely at the microscopic level, corresponding to the average size of pores or cracks in rocks, while any of the proposed macroscopic models operate on completely different (orders of magnitude larger) scales, which explains their diversity. The authors of such models simply do not have either an accurate method for describing physical processes at the microscopic level based on the fundamental laws of Newtonian continuum mechanics, or the ability to take into account the microstructure of rocks in macroscopic models. Therefore, they have to limit themselves to some speculative considerations (postulates) formulated by the authors themselves.

In view of the above, a natural question arises: if there are several macroscopic models describing the same physical process, which of them most adequately reflects this process? Where is the criterion of truth here?

The answer to this question is quite complex and is beyond the scope of this article. Lets just say that in order to derive a macroscopic model adequate to the physical process under consideration, it is first necessary, following the principles formulated in the works of J. B. Keller [17] and E. Sanchez-Palencia [18], to describe this process based on the equations of Newtons classical mechanics at the microscopic level (average size of tens of microns) and only then, using mathematically strict homogenization, derive a macroscopic model that most accurately describes this physical process.

In the present manuscript we analyze all possible structures of displacement of oil by suspension which always realized by appearance of the mixture of oil and suspension with moving boundaries, separated suspension, mixture and oil. This fact contradicts a real situation, such as the displacement of one liquid by another in a pipette in the presence of real surface tension,

given the real curvature of the droplet. (see Figure 1– Figure 5).

Figure 1: system of real pipettes for different time moments.

Figure 2: system of real pipettes for different time moments.

Figure 3: system of real pipettes for different time moments.

1.1 The problem statement.

We consider one spatial variable x and look for the solution to the Buckley–Leverett system in the domain $\Omega_T = \Omega \times (0,T)$, $\Omega = (0,1) \subset \mathbb{R} = (-\infty, \infty)$, consisting of Darcy's system of filtration

$$v_{ol} = -\frac{k}{\mu_{ol}} f_{ol}(c) \frac{\partial p_{ol}}{\partial x}, \tag{1.1}$$

$$v_{sp} = -\frac{k}{\mu_{sp}} f_{sp}(c) \frac{\partial p_{sp}}{\partial x} \tag{1.2}$$

and laws of conservation of mass

$$\frac{\partial}{\partial t}(mc) + \frac{\partial v_{ol}}{\partial x} = 0, \tag{1.3}$$

$$\frac{\partial}{\partial t}\big(m(1-c)\big) + \frac{\partial v_{sp}}{\partial x} = 0 \tag{1.4}$$

in the domain Ω_T.

The system (1.1) – (1.4) is completed with the state equations

$$kp_{ol} - kp_{sp} = p_{cap} \equiv \alpha_{cap}\, c, \tag{1.5}$$

$$f_{ol}(c) = \alpha_{ol}\, c, \;\; f_{sp}(c) = \alpha_{sp}\,(1-c) \tag{1.6}$$

and following boundary and initial conditions

$$v_{sp}(0,t) = V(t), \;\; v_{ol}(1,t) = V(t), \tag{1.7}$$

$$v_{sp}(x,0) = v_{ol}(x,0) = 0, \;\; c(x,0) = c^0(x),$$
$$c^0(x) = 0 \text{ for } 0 < x < \frac{1}{2} \text{ and } c^0(x) = 1 \text{ for } \frac{1}{2} < x < 1. \tag{1.8}$$

Figure 4: system of real pipettes for different time moments.

In (1.1) – (1.8) c is a concentration of oil in the pore liquid, $(1-c)$ is a concentration of suspension in the pore liquid, v_{ol} is the oil velocity, v_{sp} is the suspension velocity, p_{ol} is the oil pressure, p_{sp} is the suspension pressure, p_{cap} is ***Capillary Pressure***, k is the dimensionless permeability, m is the porosity, μ_{ol} is the dimensionless oil viscosity and μ_{sp} is the dimensionless suspension viscosity.

The relation (1.5) means that capillary pressure p_{cap} depends only on the concentration c of the oil.

Positive constants μ_{ol}, μ_{sp}, α_{ol}, α_{sp} and α_{cap} are supposed to be known.

First of all, using the obvious consequence of the sum of equations (1.3) and (1.4)

$$\frac{\partial}{\partial t}(v_{ol}+v_{sp})=0,\ \text{or}\ v_{ol}+v_{sp}=\pm|V(t)|, \tag{1.9}$$

we transform the equations (1.1) - (1.6) in the domain $\Omega_{mx}(t)$ to a form convenient for us:

$$\frac{f_{ol}}{\mu_{ol}}\,k\,\frac{\partial p_{ol}}{\partial x}+\frac{f_{sp}}{\mu_{sp}}\,k\,\frac{\partial p_{sp}}{\partial x}=\mp|V|, \tag{1.10}$$

$$\frac{f_{ol}}{\mu_{ol}}(k\,\frac{\partial p_{sp}}{\partial x}+\alpha_{cap}\frac{\partial c}{\partial x})+\frac{f_{sp}}{\mu_{sp}}\,k\,\frac{\partial p_{sp}}{\partial x}=\mp|V|, \tag{1.11}$$

$$k\,\frac{\partial p_{sp}}{\partial x}=\mp|V|\frac{\mu_{ol}f_{sp}}{\mu_{sp}f_{ol}+\mu_{ol}f_{sp}}-\frac{\alpha_{cap}\mu_{sp}\,f_{ol}}{\mu_{sp}f_{ol}+\mu_{ol}f_{sp}}\,\frac{\partial c}{\partial x}, \tag{1.12}$$

$$k\,\frac{\partial p_{ol}}{\partial x}=\mp|V|\frac{\mu_{sp}f_{ol}}{\mu_{sp}f_{ol}+\mu_{ol}f_{sp}}+\frac{\alpha_{cap}\mu_{ol}\,f_{sp}}{\mu_{sp}f_{ol}+\mu_{ol}f_{sp}}\,\frac{\partial c}{\partial x}, \tag{1.13}$$

$$v_{ol}=-\frac{f_{ol}}{\mu_{ol}}k\,\frac{\partial p_{ol}}{\partial x}=\pm|V|\frac{\mu_{sp}f_{ol}}{\mu_{sp}f_{ol}+\mu_{ol}f_{sp}}-\varphi(c)\frac{\partial c}{\partial x}, \tag{1.14}$$

$$v_{sp}=-\frac{f_{sp}}{\mu_{sp}}\,k\,\frac{\partial p_{sp}}{\partial x}=\pm|V|\frac{\mu_{ol}f_{sp}}{\mu_{sp}f_{ol}+\mu_{ol}f_{sp}}+\varphi(c)\frac{\partial c}{\partial x}, \tag{1.15}$$

$$\varphi(c)=\frac{\alpha_{cap}\,f_{ol}\,f_{sp}}{\mu_{sp}f_{ol}+\mu_{ol}f_{sp}}, \tag{1.16}$$

$$m\frac{\partial c}{\partial t}=\frac{\partial}{\partial x}\big(\mp|V|\frac{\mu_{sp}f_{ol}}{\mu_{sp}f_{ol}+\mu_{ol}f_{sp}}+\varphi(c)\frac{\partial c}{\partial x}\big). \tag{1.17}$$

The sign in the first summand of the right-hand side in (1.17) does not play a role because of equality

$$\frac{\partial}{\partial x}\big(\frac{\mu_{ol}f_{sp}}{\mu_{sp}f_{ol}+\mu_{ol}f_{sp}}\big)=\frac{\partial}{\partial x}\big(1-\frac{\mu_{sp}f_{ol}}{\mu_{sp}f_{ol}+\mu_{ol}f_{sp}}\big).$$

Taking into account equalities (1.6) we get

$$m\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\big(\mp |V(t)|\frac{\mu_{sp}\alpha_{ol}\, c}{\mu_{sp}f_{ol}(c) + \mu_{ol}f_{sp}(c)} + \varphi(c)\frac{\partial c}{\partial x}\big). \tag{1.18}$$

Let $S_0 = \{x \in \Omega : x = 0\}$, $S_1 = \{x \in \Omega : x = 1\}$, $\Gamma^0(t) = \{t \in (0, t_0) : x = R^0(t)\}$ be a boundary between $\Omega^0_{sp}(t) = \{x \in \Omega : 0 < x < R^0(t)\}$ and $\Omega^0_{ol}(t) = \{x \in \Omega : 0 < R^0(t) < x < 1\}$, $\Gamma^-(t) = \{t \in (0, t_0) : x = R^-(t)\}$ be the boundary between $\Omega_{sp}(t) = \{x \in \Omega : 0 < x < R^-(t)\}$, $\Omega_{mx}(t) = \{x \in \Omega : R^-(t) < x < R^+(t)\}$ and $\Gamma^+(t) = \{t \in (0, t_0) : x = R^+(t)\}$ be the boundary between $\Omega_{mx}(t)$ and $\Omega_{ol}(t) = \{x \in \Omega : R^+(t) < x < 1\}$.

We also put $\Omega^0_{ol,t_0} = \bigcup\limits_{t=0}^{t_0} \Omega^0_{ol}(t)$, $\Omega_{sp,t_0} = \bigcup\limits_{t=0}^{t_0} \Omega_{sp}(t)$, $\Omega_{mx,t_0} = \bigcup\limits_{t=0}^{t_0} \Omega_{mx}(t)$ and $\Gamma^{\pm}_{t_0} = \bigcup\limits_{t=0}^{t_0} \Gamma^{\pm}(t)$.

In this publication we will prove that the phenomenological Buckley-Leverett model does not adequately describe the physical process under consideration. To do this, we will study the problem of the decay of a discontinuity of the initial concentration of oil, when at rest in one half $\Omega_{sp}(0) = \{x \in \Omega = (0,1) : 0 < x < \frac{1}{2}\}$ of the domain Ω there is suspension, and in the other half of the domain $\Omega_{ol}(0) = \{x \in \Omega = (0,1) : \frac{1}{2} < x < 1\}$ there is an oil, and these domains are separated by an impenetrable partition $\Gamma(0) = \{x \in \Omega : x = \frac{1}{2}\}$.

At the initial moment of time, the partition is removed and between suspension in the domain $\Omega_{sp}(0)$ and oil in the domain $\Omega_{ol}(0)$ immediately appears a zone of mixing of oil and suspension. We will analyze all possible configurations of solutions for all possible boundary and initial conditions.

For the case of no mixture and the existence of only one free boundary $x = R(t)$ there are two cases.

Cnf1: the suspension in the domain $\Omega^0_{sp,t_0} = \bigcup\limits_{t=0}^{t_0} \Omega^0_{sp}(t)$, $\Omega^0_{sp}(t) = \{x \in \Omega : 0 < R(t) < x < 1\}$ displace the oil in the domain $\Omega^0_{ol,t_0} = \bigcup\limits_{t=0}^{t_0} \Omega^0_{ol}(t)$, $\Omega^0_{ol}(t) = \{x \in \Omega = (0,1) : R(t) < x < 1\}$ until to the moment t_0 when $\Omega^0_{sp}(t_0) = \Omega$.

Cnf2: the oil in the domain Ω^0_{ol,t_0} displace the suspension in the domain Ω^0_{sp,t_0} until to the moment t_0 when $\Omega^0_{ol}(t_0) = \Omega$.

We will prove that configurations *Cnf1* and *Cnf2)* are impossible.

The other configurations are only possible for two free boundaries $R^{\pm}(t)$, when between the domain $\Omega_{sp}(0) = \{x \in \Omega = (0,1) : 0 < x < \frac{1}{2}\}$, occu-

pied by suspension and domain $\Omega_{ol}(0) = \{x \in \Omega = (0,1) : \frac{1}{2} < x < 1\}$, occupied by the oil, immediately arises a zone $\Omega_{mx}(t)$ of a mixture of oil and suspension.

Here we have five different configurations.

For the sake of simplicity we put $R^-(t) = 0$ for $t > t^-$ if $R^-(t) > 0$ for $0 < t < t^-$ and $R^-(t^-) = 0$ if $R^-(t)$ is decreasing function and $R^+(t^+) = 1$ for $t > t^\pm$ if $R^+(t) < 1$ for $0 < t < t^+$ and $R^+(t^+) = 1$ if $R^+(t)$ is increasing function.

Cnf3: the mixture in the domain $\Omega_{mx,t_0} = \bigcup_{t=0}^{t_0} \Omega_{mx}(t)$, $\Omega_{mx}(t) = \{x \in \Omega = (0,1) : R^-(t) < x < R^+(t)\}$ begins to displace the suspension in the domain $\Omega_{sp,t_0} = \bigcup_{t=0}^{t_0} \Omega_{sp}(t)$, $\Omega_{sp}(t) = \{x \in \Omega = (0,1) : 0 < x < R^-(t)\}$ and oil in the domain $\Omega_{ol,t_0} = \bigcup_{t=0}^{t_0} \Omega_{ol}(t)$, $\Omega_{ol}(t) = \{x \in \Omega = (0,1) : R^+(t) < x < 1\}$ until to the moment t_0 when $\Omega_{mx}(t_0) = \Omega$;

Cnf4: the mixture in the domain Ω_{mx,t_0} begins to displace the suspension in the domain Ω_{sp,t_0} and oil in the domainΩ_{ol,t_0} until to the moment t_* and then the oil begins to display the mixture up to moment t_0 up to the moment t_0 when $\Omega_{ol}(t_0) = \Omega$;

Cnf5: the mixture in the domain Ω_{mx,t_0} begins to displace the oil in the domain Ω_{ol,t_0} and suspension in the domain Ω_{sp,t_0} until to the moment t_* and then the suspension begins to display the mixture and oil up to moment t_0 when $\Omega_{sp}(t_0) = \Omega$;

Cnf6: the oil in the domain Ω_{ol,t_0} begins to displace the mixture in the domain Ω_{mx,t_0} and suspension in the domain Ω_{sp,t_0} up to the moment t_0 when $\Omega_{ol}(t_0) = \Omega$;

Cnf7: the suspension in the domain Ω_{sp,t_0} begins to displace the mixture in the domain Ω_{mx,t_0} and oil in the domain Ω_{ol,t_0} up to the moment t_0 when $\Omega_{sp}(t_0) = \Omega$.

All these situations are realized for boundary conditions, listed below.

First two cases with one free boundary in principle be possible if $U_{sp}(0,t) = |V(t)|$, $U_{ol}(1,t) = |V(t)|$, $0 < t < t_0$ for *Cnf1* and $U_{sp}(0,t) = -|V(t)|$, $U_{ol} = -|V(t)|$, $0 < t < t_0$ *for Cnf2*.

For the case of two free boundaries there are fife different combinations of boundary conditions:

Cnf3: $U_{sp}(0,t) = -|V(t)|$, $U_{ol}(1,t) = |V(t)|$, $V_{sp} = -|V(t)|$, $V_{ol} = |V(t)|$;

Cnf4: $U_{sp}(0,t) = -|V(t)|, U_{ol}(1,t) = |V(t)|$ up to time moment t_* and then $U_{sp}(0,t) = |V(t)|, U_{ol} = |V(t)|$;

Cnf5: $U_{sp}(0,t) = -|V(t)|, U_{ol}(1,t) = |V(t)|$ up to time moment t_* and then $U_{sp}(0,t) = -|V(t)|, U_{ol} = -|V(t)|$ up to time moment t_* and then

$U_{sp}(0,t) = |V(t)|, U_{ol} = |V(t)|$;

Cnf6: $U_{sp} = -|V(t)|, U_{ol} = -|V(t)|$;

Cnf7: $U_{sp}(0,t) = |V(t)|, U_{ol}(1,t) = |V(t)|$.

The functions $R(t)$ and $R^{\pm}(t)$ satisfy the following initial condition

$$R(0) = R^{\pm}(0) = \frac{1}{2}. \tag{1.19}$$

and are assumed to be unknown.

1.2 Auxiliary statements

1.2.1 Mollifiers.

Let $J(s) \geqslant 0$, $J(s) = 0$ for $|s| > 1$, $J(s) = J(-s)$, $J \in C^{\infty}(-\infty, +\infty)$, and

$$\int_{\mathbb{R}^n} J(|\boldsymbol{x}|)dx = 1, \;\; \boldsymbol{x} \in \mathbb{R}^n, \; n > 0. \tag{1.20}$$

Definition 1.1. *The operator* $\boldsymbol{M}_h : \mathbb{L}_2(\Omega) \to \mathbb{C}^{\infty}(\overline{\Omega})$

$$\boldsymbol{M}_h < u >= u_h(\boldsymbol{x}) = \frac{1}{h}\int_{-\infty}^{\infty} J(\frac{|\boldsymbol{x}-\boldsymbol{y}|}{h})u(\boldsymbol{y})dy, \tag{1.21}$$

is called a ***mollifier****.*

Lemma 1.1. *Let* $u \in \mathbb{L}_2(\Omega)$*. Then*

$$\|u - u_h\|_{2,\Omega} \to 0 \;\; as \;\; h \to 0. \tag{1.22}$$

For proof see [19], Lemma 2.18.

1.2.2 Spaces $\mathbb{BV}(\Omega)$ and $\mathbb{L}_{\infty}\big(0,T;\mathbb{BV}(\Omega)\big)$ of functions of bounded variation. Helly's selection principle.

The present section is devoted to the Helly's selection principle [20] for the sequences of functions depending on variables x and t. The case of one variable x has been studied in [21].

Definition 1.2. *Let a function* $u(x)$ *be defined on the interval* $\Omega = (a,b)$ *and* $\Pi(n) = \{a = x_0 < x_1 < ... < x_n = b\}$ *be an arbitrary partition of* Ω *for* $n = 3, 4, ...$

Then function u *is said to be function of bounded variation if*

$$\sup_{\Pi(n)} \sum_{k=0} |u(x_{k+1}) - u(x_{k+1})| = \boldsymbol{V}(u,\Omega) < \infty. \tag{1.23}$$

Lemma 1.2. *Any bounded in $\overline{\Omega}$ monotone function $u(x)$ is a function of bounded variation. [21], §5.*

Theorem 1.1. [21],*§4 (*Helly's selection principle*)*

Let $\{u_n\}$ be a sequence bounded in $\mathbb{BV}(\Omega)$. Then there exists some subsequence, convergent everywhere in Ω.

Consequence 1.1. Any bounded in $\mathbb{BV}(\Omega)$ sequence that converges everywhere in Ω also converges in $\mathbb{L}_2(\Omega)$ ([22], chapter 10, §37).

1.3 Strong convergence criteria in $\mathbb{L}_2(\Omega_T)$.

Definition 1.3. *We say that the function $u(\boldsymbol{x},t)$, bounded in $\mathbb{L}_2(\Omega_T)$, possesses the time derivative $\dfrac{\partial u}{\partial t} \in \mathbb{L}_2\big(0,T;\mathbb{W}_2^{-1}(\Omega)\big)$, if*

$$|\int\int_{\Omega_T} u\,\frac{\partial \xi}{\partial t}dxdt| \leqslant M_u\,|\int\int_{\Omega_T} |\nabla\xi|^2 dxdt|^{\frac{1}{2}}$$

for all functions $\xi \in \mathbb{W}_2^{1,1}(\Omega_T)$ with some positive constant M_u independent of ξ.

Lemma 1.3. *Let sequences $\{u^\varepsilon\}$ and $\{\nabla u^\varepsilon\}$ are uniformly bounded in the space $\mathbb{L}_2(\Omega_T)$,and the sequence of derivatives $\{\dfrac{\partial u^\varepsilon}{\partial t}\}$ are uniformly bounded in the space $\mathbb{L}_2\big(0,T;\mathbb{W}_2^{-1}(\Omega)\big)$.*

Then there exists some subsequence of the sequence $\{u^\varepsilon\}$ strongly convergent in $\mathbb{L}_2(\Omega_T)$.

To prove the statement we first use compactness lemma ([23], chapter 1, §5.2) and after that Theorem 1 ([22], chapter 10, §37).

Finally, we note that the motion of liquids with real surface tension in pipette droplets is different from that described by configurations *Cnf*3-*Cnf*7 (see Figure 1-Figure7): in the Buckley-Leverett model, the boundary $x = R^-(t)$ changes its position in time, while in the real motion of suspension droplets this boundary is a fixed line $x = \dfrac{1}{2}$ for $0 < t < t_0$.

Throughout the text we use the notations for functional spaces and norms in these spaces adopted in [23] and [24].

2 Main results.

Let us recall, that in the real motion all velocities v_{sp}, v_{mx} and v_{ol} are positive functions. That is why in the following we will assume that for configuration *Cnf*7 $V(t)$ is smooth function and

$$V(t) > \mu_{sp}^{-1}\alpha_{cap}\alpha_{sp} > 0. \tag{2.1}$$

Definition 2.1. *We say that the triple of functions* $\{c, v, p\}$ *is a weak solution to the problem* (1.1) – (1.8) *if the functions* $c, v, p \in \mathbb{W}_2^{1,0}(\Omega_{mx,t_0})$ *and satisfy the condition of the configuration Cnf1 together with the integral identity*

$$\int_0^{t_0}\int_0^{R(t)} \big(m(1-c)\frac{\partial \eta}{\partial t} + V\frac{\partial \eta}{\partial x}\big)dxdt + \int_0^{t_0}\int_{R(t)}^{1} (mc\frac{\partial \eta}{\partial t} + V\frac{\partial \eta}{\partial x})dxdt = \int_0^{t_0}\int_0^{1} (m\frac{\partial \eta}{\partial t} + V\frac{\partial \eta}{\partial x})dxdt = m\int_0^1 \big(\eta(x,t_0) - \eta(x,0)\big)dx, \quad (2.2)$$

Definition 2.2. *We say that the triple of functions* $\{c, v, p\}$ *is a weak solution to the problem* (1.1) – (1.8) *if the functions* $c, v, p \in \mathbb{W}_2^{1,0}(\Omega_{mx,t_0})$ *and satisfy the condition of the configuration Cnf2 together with the integral identity*

$$\int_0^{t_0}\int_0^{R(t)} \big(m(1-c)\frac{\partial \eta}{\partial t} - V\frac{\partial \eta}{\partial x}\big)dxdt + \int_0^{t_0}\int_{R^-(t)}^{R^+(t)} (mc\frac{\partial \eta}{\partial t} - V\frac{\partial \eta}{\partial x})dxdt = \int_0^{t_0}\int_{R^+(t)}^{1} (m\frac{\partial \eta}{\partial t} - V\frac{\partial \eta}{\partial x})dxdt = m\int_0^1 \big(\eta(x,t_0) - \eta(x,0)\big)dx. \quad (2.3)$$

Theorem 2.1. *The problem* (1.1) – (1.8) *for configurations Cnf1 and Cnf2 has no weak solutions.*

Let

$$c^-(t) = c\big(R^-(t+0,t\big), \;\; c^+(t) = c\big(R^+(t-0,t\big), \quad (2.4)$$

where $R^-(t+0,t) = \lim\limits_{\tau\to t} R(\tau)$ and $R(\tau) > R(t)$, $R^+(t-0,t) = \lim\limits_{\tau\to t} R(\tau)$, and $R(\tau) < R(t)$ and

$$I^1_{sp} = \int_0^{t_0}\int_0^{R^-(t)} \big(m\frac{\partial \eta}{\partial t} + V(t)\frac{\partial \eta}{\partial x}\big)dxdt + \int_0^{t_0}\int_{R^-(t)}^{R^+(t)} \big(m\frac{\partial \eta}{\partial t} + v_{sp}\frac{\partial \eta}{\partial x}\big)dxdt,$$

$$I^2_{sp} = \int_0^{t_0}\int_0^{R^-(t)} \big(\frac{\partial}{\partial t}(\eta m)\big)dxdt + \int_0^{t_0}\int_{R^-(t)}^{R^+(t)} \Big(\frac{\partial}{\partial t}\big(\eta m(1-c)\big) + \frac{\partial}{\partial x}(\eta v_{sp})\Big)dxdt,$$

$$I^3_{sp} = \int_0^{t_0}\int_{R^-(t)}^{R^+(t)} mc\big(\frac{\partial \eta}{\partial t} + v_{ol}\frac{\partial \eta}{\partial x}\big)dxdt,$$

$$I^4_{sp} = \int_0^{t_0}\int_{R^-(t)}^{R^+(t)} \big(\frac{\partial}{\partial t}(\eta mc) + \frac{\partial}{\partial x}(\eta v_{ol})\big)dxdt. \quad (2.5)$$

Definition 2.3. *We say that the set of functions* $\{c, v_{sp}, v_{mx}, v_{ol}, p_{sp}, p_{ol},\}$ *is a weak solution to the problem* (1.1) – (1.8) *if these functions belong to the space* $\mathbb{W}_2^{1,0}(\Omega_{mx,t_0})$ *and satisfy the conditions of the configuration Cnf7 together with the integral identity*

$$0 = \int_0^{t_0}\int_0^{R^-(t)} \big(m(1-c)\frac{\partial \eta}{\partial t} + v_{sp}\frac{\partial \eta}{\partial x}\big)dxdt + \int_0^{t_0}\int_{R^-(t)}^{R^+(t)} \big(m(1-c)\frac{\partial \eta}{\partial t} + v_{sp}\frac{\partial \eta}{\partial x}\big)dxdt -$$
$$\int_0^{t_0}\int_0^{R^-(t)} \Big(\frac{\partial}{\partial t}\big(\eta m(1-c)\big) + \frac{\partial}{\partial x}(\eta v_{sp})\Big)dxdt -$$
$$\int_0^{t_0}\int_{R^-(t)}^{R^+(t)} \Big(\frac{\partial}{\partial t}\big(\eta m(1-c)\big) + \frac{\partial}{\partial x}(\eta v_{sp})\Big)dxdt =$$
$$\int_0^{t_0}\int_0^{R^-(t)} \big(m\frac{\partial \eta}{\partial t} + V(t)\frac{\partial \eta}{\partial x}\big)dxdt + \int_0^{t_0}\int_{R^-(t)}^{R^+(t)} \big(m(1-c)\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(\eta v_{sp})\big)dxdt -$$
$$\int_0^{t_0}\int_{R^-(t)}^{R^+(t)} \Big(\frac{\partial}{\partial t}\big(\eta m(1-c)\big) + \frac{\partial}{\partial x}(\eta v_{sp})\Big)dxdt = I^1_{sp} - I^2_{sp}.$$

Theorem 2.2. *Let*

$$V(t) > \alpha_{cap}\alpha_{sp}\mu_{sp}^{-1} = V_0 = const > 0 \ \textit{for all}\ 0 \leqslant t \leqslant t_0. \qquad (2.6)$$

Then the problem (1.1) – (1.8) *for the configuration Cnf7 has an unique weak solution.*

3 Proof of Theorem 2.1

Let smooth functions c, v^0_{sp}, v^0_{ol} in the domains Ω^0_{sp,t_0} and Ω^0_{ol,t_0} be solution to the problem and η be some arbitrary function, vanishing at the lateral boundaries S_0 and S_1.

Then for the configuration *Cnf1* $c = 0$, $v^0_{sp} = V(t)$ in the domain and hold true the integral identity

$$0 = \int_0^{t_0}\int_0^{R(t)} \eta\Big(\frac{\partial}{\partial t}\big(m(1-c)\big) + \frac{\partial}{\partial x}V\Big)dxdt + \int_0^{t_0}\int_{R(t)}^{1} \eta\Big(\frac{\partial}{\partial t}(mc) + \frac{\partial}{\partial x}V\Big)dxdt +$$
$$\int_0^{t_0}\int_0^{R(t)} \big(m(1-c)\frac{\partial \eta}{\partial t} + V\frac{\partial \eta}{\partial x}\big)dxdt + \int_0^{t_0}\int_{R(t)}^{1} (mc\frac{\partial \eta}{\partial t} + V\frac{\partial \eta}{\partial x})dxdt - I^0 =$$
$$\int_0^{t_0}\int_0^{1} (m\frac{\partial \eta}{\partial t} + V\frac{\partial \eta}{\partial x})dxdt - I^0 = 0,$$

where

$$I^0 = \int_0^{t_0}\int_0^{R(t)} \frac{\partial}{\partial t}(\eta m)dxdt + \int_0^{t_0}\int_{R(t)}^1 \frac{\partial}{\partial t}(\eta m)dxdt = \\ m\int_0^{t_0}\int_0^1 \frac{\partial \eta}{\partial t}dxdt = m\int_0^1 \big(\eta(x,t_0)-\eta(x,0)\big)dx.$$

Thus

$$\int_0^{t_0}\int_0^1 (m\frac{\partial \eta}{\partial t} + V\frac{\partial \eta}{\partial x})dxdt = \int_0^1 \big(\eta(x,t_0)-\eta(x,0)\big)dx. \qquad (3.1)$$

The configuration *Cnf2* with $c = 0$, $v^0_{sp} = -V(t)$ in the domain $\Omega^0_{sp,t_0}$and $c = 1$ $v^0_{ol} = -V(t)$ in the domain Ω^0_{ol,t_0} is treated in a similar way and expressed as an integral identity

$$\int_0^{t_0}\int_0^1 (m\frac{\partial \eta}{\partial t} - V\frac{\partial \eta}{\partial x})dxdt = \int_0^1 \big(\eta(x,t_0)-\eta(x,0)\big)dx, \qquad (3.2)$$

which contradicts an arbitrary choice of the function η and together with the integral identity (3.1) leads to the impossibility of the first two configurations.

4 Proof of Theorem 2.2

For simplicity, we restrict ourselves to the configuration *Cnf7* with two free boundaries $x = R^{\pm}(t)$, $0 \leqslant R^-(t) \leqslant R^+(t) \leqslant 1$. All other configurations are treated similarly.

4.1 Derivation of boundary conditions on the free boundary $\Gamma^-(t)$.

Let η be some arbitrary smooth function vanishing at S_0, $\Gamma^+(t)$, $t = 0$ and $t = t_0$, $V(t)$ be smooth positive function with respect to time, c be smooth function in the domain Ω_{mx,t_0}, $0 < c^-(t) \leqslant c \leqslant c^+(t) < 1$, $v_{sp}(t) = V$, $v_{ol}(t) = 0$, $c = 0$ in the domain Ω_{sp,t_0}, $v_{sp} = 0$, $v_{ol} = V(t)$, $c = 1$ in the domain Ω_{ol,t_0} and v_{ol} and v_{sp} be smooth functions defined in the domain Ω_{mx,t_0} by relations (1.14) and (1.15):

$$v_{ol} = V\frac{\mu_{sp}f_{ol}}{\mu_{sp}f_{ol}+\mu_{ol}f_{sp}} - \varphi(c)\frac{\partial c}{\partial x} = \\ f_{ol}\frac{V\mu_{sp}-\alpha_{cap}\alpha_{sp}}{\mu_{sp}f_{ol}+\mu_{ol}f_{sp}} + f_{ol}\frac{\alpha_{cap}\alpha_{sp}}{\mu_{sp}f_{ol}+\mu_{ol}f_{sp}}c\frac{\partial c}{\partial x} \geqslant 0 \\ \text{for all } 0 \leqslant t \leqslant t_0, \quad (4.1)$$

$$v_{sp} = V\frac{\mu_{ol}f_{sp}}{\mu_{sp}f_{ol}+\mu_{ol}f_{sp}} + \varphi(c)\frac{\partial c}{\partial x} =$$
$$f_{sp}\frac{V\mu_{ol}}{\mu_{sp}f_{ol}+\mu_{ol}f_{sp}} + f_{sp}\frac{\alpha_{cap}\alpha_{ol}}{\mu_{sp}f_{ol}+\mu_{ol}f_{sp}}c\frac{\partial c}{\partial x} \geqslant 0$$
$$\text{for all } 0 \leqslant t \leqslant t_0. \quad (4.2)$$

Let also

$$I^1_{sp} = \int_0^{t_0}\int_{R^-(t)}^{R^+(t)} \big(m(1-c)\frac{\partial \eta}{\partial t} + v_{sp}\frac{\partial \eta}{\partial x}\big)dxdt,$$
$$I^2_{sp} = \int_0^{t_0}\int_{R^-(t)}^{R^+(t)} \Big(\frac{\partial}{\partial t}\big(\eta m(1-c)\big) + \frac{\partial}{\partial x}(\eta v_{sp})\Big)dxdt. \quad (4.3)$$

Then we have

$$0 = \int_0^{t_0}\int_{R^-(t)}^{R^+(t)} \eta\big(\frac{\partial}{\partial t}m(1-c) + \frac{\partial v_{sp}}{\partial x}\big)dxdt +$$
$$\int_0^{t_0}\int_{R^-(t)}^{R^+(t)} \big(m(1-c)\frac{\partial \eta}{\partial t} + v_{sp}\frac{\partial \eta}{\partial x}\big)dxdt -$$
$$\int_0^{t_0}\int_{R^-(t)}^{R^+(t)} \Big(\frac{\partial}{\partial t}\big(\eta m(1-c)\big) + \frac{\partial}{\partial x}(\eta v_{sp})\Big)dxdt =$$
$$\int_0^{t_0}\int_{R^-(t)}^{R^+(t)} \big(m(1-c)\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(\eta v_{sp})\big)dxdt -$$
$$\int_0^{t_0}\int_{R^-(t)}^{R^+(t)} \Big(\frac{\partial}{\partial t}\big(\eta m(1-c)\big) + \frac{\partial}{\partial x}(\eta v_{sp})\Big)dxdt = I^1_{sp} - I^2_{sp}.$$

By construction $I^2_{sp} = 0$ due to an arbitrary choice of the function η. Therefore the equality $I^2_{sp} = 0$ implies $I^1_{sp} = 0$, or

$$I^1_{sp} = 0,\ I^2_{sp} = 0. \quad (4.4)$$

The first identity $I^1_{sp} = 0$ in (4.4) with some additional conditions can serve as a definition of the weak solution of the problem (1.1)-(1.8), and the second identity $I^2_{sp} = 0$ in (4.4) gives us the equation for finding the unknown function $R^-(t)$.

Indeed, let η be smooth function in Ω_{sp,t_0} vanishing at S_0, $t = t_0$, $t = 0$ and at $\Gamma^+(t)$ and $\eta^-(t) = \eta\big(R^-(t), t\big)$, $v^-_{sp} = v_{sp}\big(R^-(t) + 0, t\big)$. Then equality

$I^2_{sp} = 0$ results

$$0 = I^2_{sp} = \int_0^{t_0} \int_{R^-(t)}^{R^+(t)} \Big(\frac{\partial}{\partial t}\big(m\eta(1-c)\big) + \frac{\partial}{\partial x}(\eta v_{sp})\Big) dxdt = \int_0^{t_0} \big(\frac{d}{dt} \int_{R^-(t)}^{R^+(t)} m\eta(1-c)dx\big)dt + \eta^-\big(m(1-c^-)\frac{dR^-}{dt} - v^-_{sp}\big)dt = \int_0^{t_0} \eta^-\big((1-c^-)m\frac{dR^-}{dt} - v^-_{sp})\big)dt.$$

Because of the arbitrary choice of function η last equality implies

$$\big(1-c^-(t)\big)m\frac{dR^-}{dt}(t) = v^-_{sp}(t) = v_{sp}\big(R^-(t+0), t\big). \tag{4.5}$$

To find the function $c^-(t) = c\big(R^-(t)+0, t\big)$ we put

$$I^3_{sp} = \int_0^{t_0} \int_{R^-(t)}^{R^+(t)} \big(mc\frac{\partial \eta}{\partial t} + v_{ol}\frac{\partial \eta}{\partial x}\big)dxdt,$$

$$I^4_{sp} = \int_0^{t_0} \int_{R^-(t)}^{R^+(t)} \big(\frac{\partial}{\partial t}(\eta mc) + \frac{\partial}{\partial x}(\eta v_{ol})\big)dxdt \tag{4.6}$$

and consider the integral identity

$$0 = \int_0^{t_0} \int_{R^-(t)}^{R^+(t)} \eta\big(\frac{\partial}{\partial t}(mc) + \frac{\partial v_{ol}}{\partial x}\big)dxdt + \int_0^{t_0} \int_{R^-(t)}^{R^+(t)} \Big((mc)\frac{\partial \eta}{\partial t} + v_{ol}\frac{\partial \eta}{\partial x}\big)dxdt - \int_0^{t_0} \int_{R^-(t)}^{R^+(t)} \big(\frac{\partial}{\partial t}(\eta mc) + \frac{\partial}{\partial x}(\eta m v_{ol})\big)dxdt = I^3_{ol} - I^4_{ol},$$

where η vanishing at $\Gamma^+(t)$, $x = 0$ $t = 0$ and $t = t_0$.

As before (see equalities (4.4)) we conclude that

$$I^3_{sp} = 0, I^4_{sp} = 0, \tag{4.7}$$

where the first identity is the basis for the definition of the weak solution of the problem (1.1)-(1.8), and the second identity gives us the equation for finding the unknown function $c^-(t)$.

In fact, let η be the same as before and $c = 0$, $v_{ol} = 0$, $v_{sp} = V(t)$ in

Ω_{sp,t_0}. Then the second equality in (4.7) gives us

$$0 = I^4_{sp} = \int_0^{t_0}\int_{R^-(t)}^{R^+(t)} \big(\frac{\partial}{\partial t}(\eta mc) + \frac{\partial}{\partial x}(\eta v_{ol})\big)dxdt =$$

$$\int_0^{t_0} \Big(\frac{d}{dt}\int_{R^-(t)}^{R^+(t)} \eta mc dx + \eta\big(R^-(t),t\big)\big(m\frac{dR^-}{dt}(t)c^-(t) - v_{ol}\big(R^-(t)+0,t\big)\Big)dt =$$

$$\int_0^{t_0} \eta\big(R^-(t),t\big)\big(m\frac{dR^-}{dt}(t)c^-(t) - v_{ol}\big(R^-(t)+0,t\big)\Big)dt.$$

Thus, due to the arbitrary choice of functions η

$$c^-(t)m\frac{dR^-}{dt}(t) = v_{ol}\big(R^-(t)+0,t\big). \tag{4.8}$$

The sum of equalities (4.5) and (4.8) gives us

$$m\frac{dR^-}{dt}(t) = V(t). \tag{4.9}$$

Substitution (4.9) into (4.8) results

$$c^-(t) = \big(V(t)\big)^{-1}\big(R^-(t)+0,t\big) > V_0^{-1} > 0. \tag{4.10}$$

4.2 Derivation of boundary conditions for strong discontinuities at free boundary $\Gamma^+(t)$.

Let η be some arbitrary smooth function vanishing at $\Gamma^-(t)$, S_1, $t = 0$ and $t = t_0$, $V(t)$ be smooth function with respect to time, c be smooth function in the domain Ω_{mx,t_0}, $v_{sp}(t) = 0$, $v_{ol}(t) = V(t)$, $c = 1$ in the domain Ω_{ol,t_0}, v_{ol} and v_{sp} be smooth functions defined in the domain Ω_{mx,t_0} by relations (4.1) and (4.2). Let also

$$I^1_{ol} = \int_0^{t_0}\int_{R^-(t)}^{R^+(t)} \big((mc)\frac{\partial\eta}{\partial t} + v_{ol}\frac{\partial\eta}{\partial x}\big)dxdt,$$

$$I^2_{ol} = \int_0^{t_0}\int_{R^-(t)}^{R^+(t)} \big(\frac{\partial}{\partial t}(\eta mc) + \frac{\partial}{\partial t}(\eta v_{ol})\big)dxdt. \tag{4.11}$$

Then we have

$$0 = \int_0^{t_0}\int_{R^-(t)}^{R^+(t)} \eta\big(\frac{\partial}{\partial t}(mc) + \frac{\partial v_{ol}}{\partial x}\big)dxdt +$$

$$\int_0^{t_0}\int_{R^-(t)}^{R^+(t)} \big(mc)\frac{\partial\eta}{\partial t} + v_{ol}\frac{\partial\eta}{\partial x}\big)dxdt -$$

$$\int_0^{t_0}\int_{R^-(t)}^{R^+(t)} \big(\frac{\partial}{\partial t}(\eta mc) + \frac{\partial}{\partial x}(\eta v_{ol})\big)dxdt.$$

As before, we conclude that

$$I^1_{ol} = 0,\ I^2_{ol} = 0. \tag{4.12}$$

The last identity in (4.12) results

$$0 = I^2_{ol} = \int_0^{t_0}\int_{R^-(t)}^{R^+(t)} \big(\frac{\partial}{\partial t}(\eta mc) + \frac{\partial}{\partial x}(\eta v_{ol})\big)dxdt =$$
$$\int_0^{t_0}\Big(\frac{d}{dt}\int_{R^-(t)}^{R^+(t)}(\eta mc)dx - \eta\big(R^+(t),t\big)m\frac{dR^+}{dt}(t)c^+(t) + v_{ol}\big(R^+(t)-0,t\big)\Big)dt,$$

or

$$c^+(t)m\frac{dR^+}{dt}(t) = v_{ol}\big(R^+(t)-0,t\big). \tag{4.13}$$

Next we put

$$I^3_{ol} = \int_0^{t_0}\int_{R^-(t)}^{R^+(t)} \big(m(1-c)\frac{\partial\eta}{\partial t} + v_{sp}\frac{\partial\eta}{\partial x}\big)dxdt,$$
$$I^4_{ol} = \int_0^{t_0}\int_{R^-(t)}^{R^+(t)} \Big(\frac{\partial}{\partial t}\big(\eta m(1-c)\big) + \frac{\partial}{\partial x}(\eta v_{sp})\Big)dxdt \tag{4.14}$$

and consider the integral identity

$$\int_0^{t_0}\int_{R^-(t)}^{R^+(t)} \eta\big(\frac{\partial}{\partial t}m(1-c) + \frac{\partial v_{sp}}{\partial t}\big)dxdt +$$
$$\int_0^{t_0}\int_{R^-(t)}^{R^+(t)} \big(m(1-c)\frac{\partial\eta}{\partial t} + \frac{\partial}{\partial x}(\eta v_{sp})dxdt -$$
$$\int_0^{t_0}\int_{R^-(t)}^{R^+(t)} \Big(\frac{\partial}{\partial t}\big(\eta m(1-c)\big) + \frac{\partial}{\partial x}(\eta v_{sp})\Big)dxdt = I^3_{ol} - I^4_{ol}.$$

Thus,

$$0 = I^4_{ol} = \int_0^{t_0}\int_{R^-(t)}^{R^+(t)} \Big(\frac{\partial}{\partial t}\big(\eta m(1-c)\big) + \frac{\partial}{\partial x}(\eta v_{sp})\Big)dxdt =$$
$$\int_0^{t_0}\eta^+\big(v^+_{sp} - (1-c^+)m\frac{dR^+}{dt}\big)dt.$$

The last identity gives us the missing boundary condition:

$$(1-c^+)m\frac{dR^+}{dt}(t) = v^+_{sp}(t) = v_{sp}\big(R^+(t)-0,t\big). \tag{4.15}$$

The sum of equations (4.13) and (4.15) results

$$m\frac{dR^+}{dt}(t) = V(t). \tag{4.16}$$

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